Problem Set 7

Problem 1

(a) Let

$$f_n(x) = \frac{x}{1+nx}, \quad x \in [0, \infty).$$

Then, given x,

$$f(x) = \lim_{n\to\infty} \frac{x}{1+nx} = 0.$$

Given $\epsilon > 0$, let N be any integer $> \frac{1}{\epsilon}$. For $n > N$,

$$|f_n(0) - f(0)| = 0 < \epsilon,$$

and for $x > 0$,

$$|f_n(x) - f(x)| = \frac{x}{1+nx} = \frac{1}{\frac{1}{x}+n} \le \frac{1}{n} < \epsilon.$$

Thus, $f_n(x)$ converges to $f(x)$ uniformly.

(b) Let

$$f_n(x) = \frac{\sin nx}{n}, \quad x \in (-\infty, \infty).$$

Then, given x,

$$|f(x)| = \lim_{n\to\infty} \left|\frac{\sin nx}{n}\right| \le \lim_{n\to\infty} \frac{1}{n} = 0,$$

so $f(x) = 0$. Given $\epsilon > 0$, let N be any integer $> \frac{1}{\epsilon}$. For $n > N$,

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n}\right| \le \frac{1}{n} < \epsilon.$$

Thus, $f_n(x)$ converges to $f(x)$ uniformly.

(c) Let

$$f_n(x) = \frac{nx^2}{1+nx}, \quad x \in [0, \infty).$$

Then, given x,

$$f(x) = \lim_{n\to\infty} \frac{nx^2}{1+nx} = \lim_{n\to\infty} \frac{x^2}{\frac{1}{n}+x} = x.$$

Also,

$$|f_n(x) - f(x)| = \left|\frac{nx^2}{1+nx} - x\right| = \left|\frac{nx^2 - (1+nx)x}{1+nx}\right| = \frac{x}{1+nx},$$

so $f_n(x)$ converges to uniformly to $f(x)$ just as in part (a).

(d) Let

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad x \in [0, 1].$$

Then,

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+nx^2} = 0.$$

Next, observe that

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \frac{1}{2} - 0 = \frac{1}{2}$$

for all n. This shows that for $\epsilon = \frac{1}{2}$, there is no $N > 0$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \ge N$ and $x \in [0, 1]$. In other words, $f_n(x)$ does not converge to $f(x)$ uniformly.

Problem 2

(a) For $R > 0$, let

$$f_n(x) = \frac{n}{x+n}, \quad x \in [0, R].$$

For given x, $\lim_{n\to\infty} f_n(x) = 1$. Furthermore,

$$|f_n(x) - 1| = \left|\frac{n}{x+n} - 1\right| = \frac{x}{x+n} \leq \frac{R}{n}.$$

By Theorem 22.2A, $f_n(x)$ converge uniformly.

(b) Let

$$f_n(x) = \cos\frac{x}{n}, \quad |x| < R.$$

For given x, $\lim_{n\to\infty} f_n(x) = \cos 0 = 1$. By the mean value theorem, there is some c with

$$\left|\cos\frac{x}{n} - 1\right| = \left|(-\sin c)\frac{x}{n}\right| \leq \frac{R}{n}.$$

Again, by Theorem 22.2A, $f_n(x)$ converge uniformly.

(c) Let

$$f_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \quad x \in [-1, 1].$$

We have

$$\left|\frac{x^k}{k^2}\right| \leq \frac{1}{k^2},$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $f_n(x)$ converges uniformly by Theorem 22.2B.

(d) Let

$$f_n(x) = \sum_{k=1}^{\infty} \frac{\sin nx}{x^2+n^2}, \quad x \in (-\infty, \infty).$$

We have

$$\left|\frac{\sin kx}{x^2+k^2}\right| \leq \frac{1}{x^2+k^2} \leq \frac{1}{k^2},$$

so $f_n(x)$ converge uniformly by theorem 22.2B as in (c).

Problem 3

Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Then,

$$|a_n \sin nx| \leq |a_n|$$

for $x \in (-\infty, \infty)$ and $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $(-\infty, \infty)$ by the Weierstrass M-test.

Problem 4

Let

$$u_n(x) = \frac{x}{n(x+n)}$$

for $x \in [0, \infty)$. We show that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in an interval $[0, R]$ for any $R > x$. We have

$$0 \leq u_n(x) = \frac{x}{n(x+n)} \leq \frac{R}{n^2}.$$

By Theorem 22.2B, our claim follows. Then, by Theorem 22.3, $\sum_{n=1}^{\infty} u_n(x)$ is continuous at x.

It is important to note that we did not show $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on $[0, \infty)$. This does not follow from the fact that it converges uniformly on $[0, R]$ for every R and is, in fact, false.

Problem 5

Consider the sum

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)x^n}{nx^{n-1}} \right| = \lim_{n\to\infty} \frac{n+1}{n}|x| = |x|.$$

Thus, $f(x)$ has radius of convergence 1. By Corollary 22.4, for $x \in (-1, 1)$,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{n+1}{n+1} x^{n+1} = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

Then, by the fundamental theorem of calculus,

$$f(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1 \cdot (1-x) - x \cdot (-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Problem 6

Let $P : \mathbb{R}^2 \to \mathbb{R}^3$ be

$$P(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2}\right)$$

and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{1 - P(x_1, y_1) \cdot P(x_2, y_2)}.$$

In general, if $v, w \in \mathbb{R}^n$ and $\| \cdot \|$ is the standard Euclidean norm, then $\|v - w\| = \sqrt{\|v\|^2 + \|w\|^2 - 2v.w}$, so

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{\frac{1 + 1 - 2P(x_1, y_1).P(x_2, y_2)}{2}} = \frac{1}{\sqrt{2}}\|P(x_1, y_1) - P(x_2, y_2)\|,$$

using the fact that P maps onto the unit sphere. Therefore, the three parts of this problem follow from the same statements for the Euclidean norm because P is one-to-one.

(1) For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$d\big((x_1, y_1), (x_2, y_2)\big) = \frac{1}{\sqrt{2}}\|P(x_1, y_1) - P(x_2, y_2)\| \geq 0.$$

Furthermore, $d\big((x_1, y_1), (x_2, y_2)\big) = 0$ if and only if $P(x_1, y_1) = P(x_2, y_2)$ if and only if $(x_1, y_1) = (x_2, y_2)$.

(2) Compute

$$d\big((x_1, y_1), (x_2, y_2)\big) = \frac{1}{\sqrt{2}}\|P(x_1, y_1) - P(x_2, y_2)\| = \frac{1}{\sqrt{2}}\|P(x_2, y_2) - P(x_1, y_1)\| = d\big((x_2, y_2), (x_1, y_1)\big).$$

(3) Compute

$$\begin{aligned} d\big((x_1, y_1), (x_2, y_2)\big) + d\big((x_2, y_2), (x_3, y_3)\big) &= \frac{1}{\sqrt{2}}\left(\|P(x_1, y_1) - P(x_2, y_2)\| + \|P(x_2, y_2) - P(x_3, y_3)\|\right) \\ &\geq \frac{1}{\sqrt{2}}\|P(x_1, y_1) - P(x_3, y_3)\| \\ &= d\big((x_1, y_1), (x_3, y_3)\big). \end{aligned}$$